

Cooperative oligopoly games with boundedly rational firms

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Abstract

We analyze cooperative Cournot games with externalities. Due to cognitive constraints, the members of a coalition cannot accurately predict the coalitional actions of the non-members. Thus, they compute their value following simple heuristics. In particular, they assign various non-equilibrium probability distributions over the outsiders' set of partitions. We construct the value function of a coalition and analyze the core of the corresponding games. We show that the core is non-empty provided the number of firms in the market is sufficiently large. Moreover, if two distributions over the set of partitions are related via first-order dominance, then the core of the game under the dominated distribution is a subset of the core under the dominant one. Finally, we allow the deviant coalition to act as a Stackelberg leader and show that in this case the corresponding core is empty.

Keywords: Cooperative game; externalities; Cournot market; core; bounded rationality

1 Introduction

The issue of cooperation among firms in oligopolistic markets constantly attracts the interest of economists. Traditionally, the majority of works that analyze this issue is based on non-cooperative games, as agreements among firms are often non-enforceable by any outside entity, such as a court. If however the signing of enforceable agreements is possible then the theory of cooperative games can be utilized to study cooperative market behaviors.¹ Under the last approach, economists

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¹For a discussion on legal cartels, we refer the reader to Dick (1996), Motta (2007) or Haucap et.al (2010).

usually focus on the core of an appropriately defined cooperative oligopoly game. The core consists of all allocations of total market profits that cannot be blocked by any coalition of firms. When a coalition contemplates blocking an allocation it needs to calculate its worth. In a market environment such a calculation is not a trivial task, as the coalition's worth depends on how the non-members act. Namely, it depends on the coalition structure that the outsiders will form.

Different beliefs about the reaction of the outsiders lead to different notions of core. The α and β cores (Aumann 1959) are based on min-max behavior on behalf of the non-members; the γ core (Chander & Tulkens 1997) is based on the assumption that outsiders play individual best replies to the deviant coalition. Various authors applied these core notions to the study of Cournot markets. Rajan (1989) used the concept of γ core and showed that it is non-empty for a market with 4 firms. A more general result for any number of firms is provided by Chander (2009). Currarini & Marini (1998) built a refinement of the γ core by assuming that the deviant coalition acts as a Stackelberg leader in the product market. Zhao (1999) showed that the α and β cores of oligopolistic markets are non-empty.

The seminal work of Ray & Vohra (1999) goes one step further, as the worth of a coalition is deduced via arguments that satisfy a consistency criterion: a deviant coalition takes into account the fact that after its deviation, other deviations might follow, with the newly deviant coalitions thinking in a similar forward way. For games where binding agreements are feasible, Huang & Sjostrom (1998, 2003) and Koczy (2007) developed the concept of recursive core. The recursive core is constructed under the assumption that the members of a coalition compute their value by looking recursively on the cores of the sub-games played among the outsiders.

Deducing the coalitional actions in a game with many players is computationally cumbersome. Sandholm et.al (1999) showed that for an n -player game the number of different coalition structures is $O(n^n)$ and $\omega(n^{\frac{n}{2}})$. Hence, computing the coalition structure that the outsiders form is a particularly difficult task (at least, for games with a large number of players). As a matter of fact, the problem of finding the coalition structure that maximizes the sum of all players' payoffs is *NP*-hard (Sandholm et.al 1999). Even finding sub-optimal solutions requires the search of an exponential number of cases.

The last considerations give the motivation of the current paper. We analyze an n -firm cooperative Cournot oligopoly assuming that no group of firms has the cognitive ability to accurately deduce the coalition structure that the non-members will form. As a result of this constraint, the members of a coalition cannot compute their value with precision. Instead, they compute it by following simple procedures or heuristics.

Clearly, the number of different heuristics than one can adopt is very large. Computer scientists, for example, model similar situations via search algorithms that give solutions within certain bounds from the optimal coalition structure (Sandholm et.al 1999, Dang & Jennings 2004). On the other hand, the economists' toolbox of heuristics includes models with players of various degrees of cognitive

abilities (Stahl & Wilson 1994, Camerer 2003, Camerer et.al 2004, Haruvy & Stahl 2007), models with probabilistic choice rules (McKelvey & Palfrey 1995, Chen et.al 1997, Anderson et.al 2002), to name only a few.

In our paper, the heuristics are based on the assignment of non-equilibrium probability distributions over the set of coalition structures that the opponents can form. I.e, when contemplating a deviation from the grand coalition, the members of a coalition make the simplifying assumption that the reactions of the outsiders follow various plausible -but not necessarily optimal- probability distributions. These distributions reflect various degrees of sophistication on behalf of the firms.

At a first step, we assume that the members of the deviant coalition are completely ignorant about the reactions of the non-members. Hence, they assign a uniform distribution over all possible coalition structures. Then, we assume that the probability of a coalition structure is proportional to the relative profitability that the structure induces for the outsiders. Namely, the deviant coalition assumes that it is more likely that its opponents will manage to coordinate and partition according to the more efficient structures. This last approach is in the spirit of the logit quantal response approach of McKelvey & Palfrey (1995) in non-cooperative games, where the probability of choosing a strategy depends on its relative payoff, with the probability being positive even if the strategy is inferior.

We focus on a market with linear demand and cost functions. Firms are cost-symmetric and produce a homogeneous good. We derive the value function of a coalition under the above probabilistic scenarios and we examine the core of the corresponding games. We show that if the number of firms in the market is sufficiently large then the core is non-empty under both distributions. Hence, bounded rationality results into cooperation among all firms in the market.

In the second part of the paper we extend the analysis into two directions. We first compare the cores of the above games with the core of a game constructed under a probability distribution that first-order dominates the uniform and the logit distributions. Such a distribution gives relatively high weight to partitions that consist of many coalitions. Its use is justified if we assume that the formation of large coalitions is more costly compared to the formation of small ones. We show that the core of the game under the uniform or the logit distribution is contained in the core of a game under any distribution that first-order dominates them.

In particular, the above inclusion holds for the case of γ core. Namely, the core under our distributions is contained in the core of the oligopoly game constructed under the assumption that outsiders remain separate entities (in our terminology, the γ scenario corresponds to the degenerate distribution that assigns probability one to the singletons partition). Hence our concept of core refines the γ core.

Finally, we extend the analysis by allowing the deviant coalition to assume the role of Stackelberg leader in the market. Namely, a coalition that contemplates a deviation believes its opponents will select their quantities after observing its quantity choice. As in the case of simultaneous quantity choices, the deviant coalition reasons again probabilistically over the outsiders' partitions. We show that in such an environment the core of the corresponding game is always empty,

in contrast to the simultaneous game.

In what follows, we present the basic model in section 2 and in section 3 we present our results. Section 4 provides some extensions and section 5 concludes.

2 The model

We consider a market with the set $N = \{1, 2, \dots, n\}$ of firms. Firms produce a homogeneous product. They face the inverse demand function $P = \max\{a - Q, 0\}$ where P is the market price, $Q = q_1 + q_2 + \dots + q_n$ is the market quantity, q_l is the quantity of firm l , $l = 1, 2, \dots, n$ and $a > 0$. Firm l produces with the cost function $C(q_l) = cq_l$, where $0 < c < a$.

Let $S \subset N$ denote a coalition with $|S| = s$ members and let N/S denote the complementary set of S , where $|N/S| = n - s$. The worth or value of S is the sum of its members' profits. In order to compute this value, the members of S need to predict how the members of N/S partition themselves into coalitions. The set N/S can be partitioned into disjoint subsets in B_{n-s} ways, where B_{n-s} is Bell's $(n - s)^{th}$ number (Bell 1934). The B_{n-s} different partitions define B_{n-s} different coalition structures that coalition S might face in the market.

Notice that what matters for a deviant coalition in a linear, symmetric Cournot market is the number of the opponent coalitions and not the synthesis of the coalitions. Namely, all coalition structures with j coalitions would induce the same profit for S , irrespective of how the $n - s$ outside firms are grouped among the j coalitions. We call all these structures *j-similar*. Given this similarity, we simply let \mathcal{P}_j denote a coalition structure with j coalitions (without the need to specify the exact allocation of the $n - s$ firms among the j coalitions).

Due to cognitive constraints, S uses simple probabilistic models in order to predict the coalitional behavior of the $n - s$ non-members. In this section we develop two such models, which reflect varying degrees of cognition or sophistication. Under the first, firms are completely ignorant about the outsiders' coalitional actions and treat all partitions as equiprobable. Under the second, firms are more sophisticated and assume that the probability of a partition is proportional to the profitability that the partition induces for the outsiders. The second approach is in line with the spirit of the quantal response model (McKelvey & Palfrey 1995) in non-cooperative games, where the probability of choosing a strategy depends on its relative payoff, with the probability being positive even if the strategy is inferior.

2.1 The uniform model

Denote the number of j -similar coalition structures by $K_{n-s,j}$, where $K_{n-s,j}$ gives the number of ways to partition a set of $n - s$ objects into j groups, or else the Stirling numbers of the second kind:

$$K_{n-s,j} = \left\{ \begin{matrix} n-s \\ j \end{matrix} \right\} = \frac{1}{j!} \sum_{i=0}^j (-1)^i \binom{j}{i} (j-i)^{n-s} \quad (1)$$

To model the first scenario, i.e., the scenario of complete ignorance, define the function

$$f_{n,s}(j) = \frac{K_{n-s,j}}{B_{n-s}}, \quad j = 1, 2, \dots, n-s \quad (2)$$

Notice that $f_{n,s}(j) \in (0, 1)$ and $\sum_{j=1}^{n-s} f_{n,s}(j) = 1$ (this is because the Stirling numbers of the second kind add up to the corresponding Bell number). Then, $f_{n,s}(j)$ gives the total probability that a coalition with s members assigns to all j -similar structures.

2.2 The logit model

Let us now turn to the second scenario. The members of S believe that the outside players can coordinate to partitions with probabilities which are proportional to the relative profitability of the partitions. We assume that coordination is subject to error, the error following the logit quantal response model (McKelvey & Palfrey 1995). In particular we shall adopt an approach which indirectly assumes that, from the perspective of S , all firms in N/S are one entity which decides how to break into a number of smaller entities via the logit error model.

The one-entity assumption needs elaboration. We mainly use it in order to utilize an one-person quantal response model, where a single decision maker has the strategy set $\{1, 2, \dots, n-s\}$, i.e., the number of entities in which it can break into. We could adopt an alternative approach under which the members of S model the $n-s$ outsiders as many entities, who take decisions regarding coalition formation again via the logit error model. We choose the former over the latter out of simplicity considerations.

Let Π_j denote the total profits of the j coalitions under any partition with j members. These are the total profits that j firms earn in a market with $j+1$ competing firms (the j coalitions plus S). Recall that a partition with j members forms in $K_{n-s,j}$ ways. Then we define the probability of partitions with j coalitions as

$$f_{n,s}(j) = \frac{\exp(\Pi_j) K_{n-s,j}}{\sum_{m=1}^{n-s} \exp(\Pi_m) K_{n-s,m}} \quad (3)$$

Notice that a partition of size j is weighted by its multiplicity² $K_{n-s,j}$. One problem that arises is that not all j -similar partitions are equally stable. Let for example

²In one sense, the number $K_{n-s,j}$ expresses the "easiness" with which partitions with j members can occur. The results of the paper hold even if the profitability of a partition

$N = \{1, 2, 3, 4, 5\}$, $S = \{5\}$, $N/S = \{1, 2, 3, 4\}$ and $j = 2$. Consider the partitions $\{\{1, 2\}, \{3, 4\}\}$ and $\{\{1, 2, 3\}, \{4\}\}$. On the one hand, these partitions are 2-similar. On the other, the stability of the two partitions is not the same: assuming an equal division of profits, firm 4 prefers the latter over the former partition. So, grouping them together would seem problematic. We bypass this problem by assuming that intra-coalitional transfers are possible. This last assumption is itself justified by interpreting N/S as one entity.³

2.3 Value function

Let us now compute the value function of a coalition. A significant portion of the analysis that follows does not depend on whether we use (2) or (3). To give a unified representation let $\lambda \in \{0, 1\}$. Define

$$f_{n,s}^\lambda(j) = \frac{\exp(\lambda \Pi_j) K_{n-s,j}}{\sum_{m=1}^{n-s} \exp(\lambda \Pi_m) K_{n-s,m}} \quad (4)$$

When $\lambda = 0$, we obtain (2) and when $\lambda = 1$ we obtain (3). To compute the worth of S we can use the j -similarity and focus for each j on one representative of the similar structures. So, let $q_i^{\mathcal{P}_j}$ denote the quantity that coalition i chooses, $i = 1, 2, \dots, j$, under structure \mathcal{P}_j ; and let q_s denote the quantity of coalition S . The profit function that coalition S faces is then given by

$$\pi(S) = \sum_{j=1}^{n-s} f_{n,s}^\lambda(j) (a - q_s - \sum_{i=1}^j q_i^{\mathcal{P}_j} - c) q_s \quad (5)$$

The profit function of coalition i under structure \mathcal{P}_j , $j = 1, 2, \dots, n - s$, is

$$\pi_i^j = (a - q_s - \sum_{r=1, r \neq i}^j q_r^{\mathcal{P}_j} - q_i^{\mathcal{P}_j} - c) q_i^{\mathcal{P}_j}, \quad i = 1, 2, \dots, j$$

Hence the maximization problems to solve for are

$$\max_{q_s} \pi(S)$$

and for $j = 1, 2, \dots, n - s$,

$$\max_{q_i} \pi_i^j, \quad i = 1, 2, \dots, j$$

with j members is not adjusted by $K_{n-s,j}$.

³This issue does not arise in subsection 2.1, due to the total naiveness of players there.

By symmetry, the solution of the above problems will involve $q_1^{\mathcal{P}_j} = q_2^{\mathcal{P}_j} = \dots = q_j^{\mathcal{P}_j}$. Define

$$F_{f_{n,s}^\lambda} = \sum_{j=0}^{n-s} \frac{j \cdot f_{n,s}^\lambda(j)}{j+1} \quad (6)$$

Then it is easy to show that the solution of the maximization problems is given by

$$q_s = \frac{1 - F_{f_{n,s}^\lambda}}{2 - F_{f_{n,s}^\lambda}}(a - c) \quad (7)$$

and for $j = 1, 2, \dots, n - s$,

$$q_i^{\mathcal{P}_j} = \frac{a - c}{(j+1)(2 - F_{f_{n,s}^\lambda})}, \quad i = 1, 2, \dots, j \quad (8)$$

Using (7) and (8) in (5), we obtain the worth function $v(S)$ as⁴

$$v(S) = (a - c)^2 \frac{1 - F_{f_{n,s}^\lambda}}{(2 - F_{f_{n,s}^\lambda})^2} \sum_{j=0}^{n-s} \frac{f_{n,s}^\lambda(j)}{j+1} \quad (9)$$

Hence our game is the pair (N, v) where v is defined by (9).

3 Results

With a slight abuse of notation, let $v^n(s)$ denote the worth of a coalition with s members in a game with n players.

Lemma 1 *Let $\lambda \in \{0, 1\}$. For every positive integer k , the equality $v^n(s) = v^{n+k}(s+k)$ holds.*

Proof. Consider an outsiders' partition consisting of j members. Notice that the total profits of the j coalitions do not depend on the number of members of the deviant coalition S . Hence

$$f_{n,s}^\lambda(j) = \frac{\exp(\lambda \Pi_j) K_{n-s,j}}{\sum_{m=1}^{n-s} \exp(\lambda \Pi_m) K_{n-s,m}} =$$

⁴For notational simplicity, we do not include a λ in the notation of the value function. Moreover, we should sum from $j = 1$ up to $n - s$ but for convenience we start from $j = 0$ with the understanding that $f_{n,s}^\lambda(0) = 0$.

$$\frac{\exp(\lambda \Pi_j) K_{n+k-(s+k),j}}{\sum_{m=1}^{n+k-(s+k)} \exp(\lambda \Pi_m) K_{n+k-(s+k),m}} = f_{n+k,s+k}^\lambda(j) \quad (10)$$

Expression (10) implies that $F_{f_{n,s}^\lambda} = F_{f_{n+k,s+k}^\lambda}$. This combined with (9) proves the result. \blacksquare

The intuition behind Lemma 1 is clear. Let S_{+k} denote a deviant coalition that has $s + k$ players in a game with $n + k$ players. Then S_{+k} faces the number $n + k - s - k = n - s$ of outsiders, which is equal to the number of outsiders that a coalition with s members faces in a game with n players. Hence the two coalitions face the same set of potential coalition structures and thus have the same value.

An almost immediate implication of Lemma 1 is the monotonicity of $v^n(S)$ in $s = |S|$.

Lemma 2 *Let $\lambda \in \{0, 1\}$. For every n , $v^n(s)$ is strictly increasing in s .*

Proof. We will use induction on the number of players, n . For the base case, $n = 2$, we have to prove that $v^2(2) > v^2(1) > v^2(0)$. Notice that $v^2(2) = \left(\frac{a-c}{2}\right)^2 > \left(\frac{a-c}{3}\right)^2 = v^2(1) > 0 = v^2(0)$, so we have the base case (irrespective of $\lambda \in \{0, 1\}$).

Assume for the induction hypothesis that in a game with n players and for an arbitrary s , $1 < s \leq n$ we have that $v^n(s) > v^n(s-1)$.

We will prove that $v^{n+1}(s) > v^{n+1}(s-1)$. But this is an immediate result of lemma 1 and the induction hypothesis since $v^{n+1}(s) = v^n(s-1) > v^n(s-2) = v^{n+1}(s-1)$. \blacksquare

Proposition 1 *Let $\lambda \in \{0, 1\}$. The game (N, v) has a non-empty core if n is sufficiently large.*

Proof Since firms are identical, the core is non-empty if and only if for all S : $|S| = s \leq n$,

$$\frac{v^n(n)}{n} \geq \frac{v^s(s)}{s} \quad (11)$$

It is easy to verify that the inequality does not hold for $3 \leq n \leq 10$, irrespective of $\lambda \in \{0, 1\}$. So for these values of n the core is empty.⁵ The inequality holds for $n = 11$ for all $\lambda \in \{0, 1\}$. (Table 1 in the Appendix). We will prove the rest of the proposition using induction on n , $n \geq 11$.

Base: Table 1 in the Appendix establishes the base case ($n = 11$).

⁵For $3 \leq n \leq 10$ it holds that $v^n(1) > \frac{v^n(n)}{n}$. See Table 2 in the Appendix. The relevant calculations were made using the Maple program and they are available by the authors upon request.

Induction hypothesis: For all $S : |S| = s \leq n$, $\frac{v^n(n)}{n} \geq \frac{v^n(s)}{s}$.

Induction step: We will show that for all $S : |S| = s \leq n + 1$,

$$\frac{v^{n+1}(n+1)}{n+1} \geq \frac{v^{n+1}(s)}{s}$$

By Lemma 1 we have that $v^{n+1}(s) = v^{n+1}((s-1)+1) = v^n(s-1)$ and also that $v^{n+1}(n+1) = v^n(n)$. So we have to show that

$$\frac{v^n(n)}{n+1} \geq \frac{v^n(s-1)}{s} \quad (12)$$

From the Induction hypothesis we have

$$v^n(n) \geq \frac{n}{s-1} v^n(s-1)$$

and thus

$$(s-1)v^n(n) \geq nv^n(s-1) \quad (13)$$

Using Lemma 2,

$$v^n(n) > v^n(s-1) \quad (14)$$

Adding (13) and (14) we have

$$sv^n(n) > (n+1)v^n(s-1)$$

which implies that (12) holds. So we have the proof for $n+1$ and thus the proposition. \blacksquare

Our analysis shows that the core is non-empty provided the number of firms n is sufficiently large. In other words, for low values of n , the sign of $v^n(n)/n - v^n(s)/s$ can be negative for some s , whereas for large n this sign is always positive: as n increases both $v^n(n)/n$ and $v^n(s)/s$ decrease. However, the second term decreases faster than the first. As a result, for sufficiently large n the difference $v^n(n)/n - v^n(s)/s$ becomes positive for all s and the core is non-empty.

4 Extensions

4.1 A representation of $v(S)$

Let us give a useful representation of $v(S)$ that will be used later on. The representation is based on harmonic numbers. Recall that the k -th harmonic number is defined as

$$h^k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} = \sum_{j=0}^{k-1} \frac{1}{1+j} \quad (15)$$

where k is a natural number. Let us now define

$$h_f^k = \sum_{j=0}^{k-1} \frac{f(j)}{1+j} \quad (16)$$

as the k -th probabilistic harmonic number induced by f , where $f(\cdot)$ is a probability distribution on $\{0, 1, 2, \dots, k\}$. To make a connection between this concept and our game, notice that we can write (24) as

$$v(S) = \frac{1 - F_{f_{n,s}}}{(2 - F_{f_{n,s}})^2} (a - c)^2 h_{f_{n,s}}^{n-s+1} \quad (17)$$

where $h_{f_{n,s}}^{n-s+1} = \sum_{j=0}^{n-s} \frac{f_{n,s}(j)}{1+j}$. Notice next that

$$F_{f_{n,s}} = \sum_{j=0}^{n-s} \frac{j \cdot f_{n,s}(j)}{j+1} = \sum_{j=0}^{n-s} \left[1 - \frac{1}{j+1}\right] f_{n,s}(j) = 1 - \sum_{j=0}^{n-s} \frac{f_{n,s}(j)}{j+1}$$

Hence

$$F_{f_{n,s}} = 1 - h_{f_{n,s}}^{n-s+1} \quad (18)$$

Then combining (17) and (18) we get

$$v(S) = \frac{(h_{f_{n,s}}^{n-s+1})^2}{(1 + h_{f_{n,s}}^{n-s+1})^2} (a - c)^2 \quad (19)$$

Representations of the form (19) hold for any probability distribution that a coalition assigns over the set of coalition structures. So consider two such distributions $g_{n,s}$ and $z_{n,s}$. Let (N, v_g) and (N, v_z) denote the corresponding games and let C_g and C_z denote the cores of the two games. We have the following:

Lemma 3 *Consider two probability distributions $g_{n,s}$ and $z_{n,s}$ such that $h_{g_{n,s}}^{n-s+1} > h_{z_{n,s}}^{n-s+1}$, for all s . If $C_g \neq \emptyset$ then $C_z \neq \emptyset$ as well.*

Proof Since $h_{g_{n,s}}^{n-s+1} > h_{z_{n,s}}^{n-s+1}$ then $v_g(S) > v_z(S)$. Let $x \in C_g$. Then $\sum_{i \in S} x_i \geq v_g(S) > v_z(S)$ for any S . Hence $x \in C_z$ and $C_z \neq \emptyset$. \blacksquare

We can re-state the above result by saying that if $h_{g_{n,s}}^{n-s+1} > h_{z_{n,s}}^{n-s+1}$ then $C_g \subseteq C_z$ (provided that $C_z \neq \emptyset$). Lemma 3 is useful as it provides a simple method to compare the cores of two different games. If we know that the core of one of the two games is non-empty, Lemma 3 gives us a (computationally efficient) sufficient condition for the non-emptiness of the core of the other: we simply need to compare two numbers, i.e., the probabilistic harmonic numbers induced by the corresponding distributions.

4.2 First-order dominance

Let $z_{n,s}$ be a probability distribution that the members of S assign to the set of all possible coalition structures of the non-members. Assume that $z_{n,s}$ dominates $f_{n,s}^\lambda$ at first order, i.e., $\sum_{j \leq j^*} z_{n,s}(j) \leq \sum_{j \leq j^*} f_{n,s}^\lambda(j)$, for all j^* .

Compared to the cumulative distribution of $f_{n,s}^\lambda$, the cumulative distribution of $z_{n,s}$ assigns higher probabilities to events that include coalition structures with many coalitions. Such a distribution is plausible if we accept the assumption that a structure with few coalitions is in general less likely to form (few coalitions means that the $n-s$ players form large coalitions; and large coalitions require more effort, time, coordination, etc).⁶

Denote by $v_z(S)$ the worth of S under $z_{n,s}$. Let C_z denote the core of the resulting game (N, v_z) and let C_{f^λ} denote the core of the game under (4).

Proposition 2 *Assume that $z_{n,s}$ dominates $f_{n,s}^\lambda$ at first order. The game (N, v_z) has non-empty core if n is sufficiently large. Furthermore, $C_{f^\lambda} \subseteq C_z$, for all $\lambda \in \{0, 1\}$.*

Proof We shall use Proposition 1 and Lemma 3. First we show that the probabilistic harmonic number induced by $f_{n,s}^\lambda$ is larger than that induced by $z_{n,s}$. To see this, notice that $h_{f_{n,s}^\lambda}^{n-s+1} > h_{z_{n,s}}^{n-s+1}$ iff

$$\sum_{j=0}^{n-s} \frac{1}{1+j} [f_{n,s}^\lambda(j) - z_{n,s}(j)] > 0 \quad (20)$$

Since $f_{n,s}^\lambda(0) = z_{n,s}(0) = 0$, we have

$$\begin{aligned} \sum_{j=0}^{n-s} \frac{1}{1+j} [f_{n,s}^\lambda(j) - z_{n,s}(j)] &= \sum_{j=1}^{n-s} \frac{1}{1+j} [f_{n,s}^\lambda(j) - z_{n,s}(j)] = \\ &\overbrace{\frac{f_{n,s}^\lambda(1) - z_{n,s}(1)}{2}}^{\geq 0} + \frac{f_{n,s}^\lambda(2) - z_{n,s}(2)}{3} + \sum_{j=3}^{n-s} \frac{1}{1+j} [f_{n,s}^\lambda(j) - z_{n,s}(j)] > \\ &\frac{1}{3} \overbrace{\left[\sum_{j=1}^2 f_{n,s}^\lambda(j) - \sum_{j=1}^2 z_{n,s}(j) \right]}^{\geq 0} + \sum_{j=3}^{n-s} \frac{1}{1+j} [f_{n,s}^\lambda(j) - z_{n,s}(j)] = \\ &\frac{1}{3} \overbrace{\left[\sum_{j=1}^2 f_{n,s}^\lambda(j) - \sum_{j=1}^2 z_{n,s}(j) \right]}^{\geq 0} + \frac{f_{n,s}^\lambda(3) - z_{n,s}(3)}{4} + \sum_{j=4}^{n-s} \frac{1}{1+j} [f_{n,s}^\lambda(j) - z_{n,s}(j)] > \end{aligned}$$

⁶The cost of forming coalitions is not modeled here. It is simply reflected in the cumulative distribution of a probability scheme.

$$\frac{1}{4} \overbrace{\left[\sum_{j=1}^3 f_{n,s}^\lambda(j) - \sum_{j=1}^3 z_{n,s}(j) \right]}^{\geq 0} + \sum_{j=4}^{n-s} \frac{1}{1+j} [f_{n,s}^\lambda(j) - z_{n,s}(j)]$$

Continuing the iterations, we eventually get that

$$\begin{aligned} \sum_{j=0}^{n-s} \frac{1}{1+j} [f_{n,s}^\lambda(j) - z_{n,s}(j)] &> \frac{1}{n-s} \overbrace{\left[\sum_{j \leq n-s-1} f_{n,s}^\lambda(j) - \sum_{j \leq n-s-1} z_{n,s}(j) \right]}^{\geq 0} + \\ \frac{f_{n,s}^\lambda(n-s) - z_{n,s}(n-s)}{1+n-s} &= \\ \frac{1}{n-s} \overbrace{\left[\sum_{j=1}^{n-s-1} f_{n,s}^\lambda(j) - \sum_{j=1}^{n-s-1} z_{n,s}(j) \right]}^{\geq 0} &+ \frac{1}{1+n-s} \left[- \sum_{j=1}^{n-s-1} f_{n,s}^\lambda(j) + \sum_{j=1}^{n-s-1} z_{n,s}(j) \right] = \\ \overbrace{\left[\sum_{j \leq n-s-1} f_{n,s}^\lambda(j) - \sum_{j \leq n-s-1} z_{n,s}(j) \right]}^{\geq 0} \left(\frac{1}{n-s} - \frac{1}{1+n-s} \right) &> 0 \end{aligned}$$

So, (20) is proved. Furthermore, by Proposition 1 we know that $C_f \neq \emptyset$ whenever n is large ($n \geq 11$). Hence by Lemma 3, $C_z \neq \emptyset$ and $C_f \subseteq C_z$. \blacksquare

Under $z_{n,s}$, the core is non-empty more often. The reason is that in comparison to $f_{n,s}^\lambda$, distributions like $z_{n,s}$ "penalize" the structures that are more favorable for a deviant coalition (i.e, structures with few coalitions) and give more weight to less favorable structures (i.e., structures with many coalitions).

A particular probability distribution that dominates $f_{n,s}^\lambda$ is the distribution defined by $\tilde{z}_{n,s}(j) = 0$, for $j = 1, 2, \dots, n-s-1$ and $\tilde{z}_{n,s}(n-s) = 1$. This degenerate distribution corresponds to the γ core scenario. It is known that the latter core is non-empty for Cournot oligopolies (Chander 2009). Notice that $h_{\tilde{z}_{n,s}^{n-s+1}} = \frac{1}{1+n-s}$ and hence $h_{f_{n,s}^{n-s+1}} > h_{\tilde{z}_{n,s}^{n-s+1}}$, as the number $h_{f_{n,s}^{n-s+1}}$ is a weighted average of the list of numbers $(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{1+n-s})$. Hence the core of (N, v) is a subset of the γ core (as an implication of Lemma 3).

The γ core is based on the worst scenario for the members of the deviant coalition S : all $n-s$ firms remain separate entities. Under $f_{n,s}^\lambda$, the singleton coalitions structure is one only of the structures that the members of S take into account. Other, more favorable, structures occur with positive probability. Hence, the worth of S in our game is no less than its worth under the γ theory, which explains the relation between the two cores.

4.3 Stackelberg play

In this section we analyze a game where the deviant coalition assumes for itself the role of Stackelberg leader. I.e., should it deviate, S assumes that the outsiders will decide on their quantities after observing S 's quantity choice. This concept of coalitional deviation was first appeared in Currarini & Marini (1998) for the case of γ core. We will show that Stackelberg play on behalf of the deviant coalition in our framework produces a new result regarding core non-emptiness.

Since the members of S assume for themselves the role of leader in deciding their quantity, we first need to solve for the maximization problems

$$\max_{q_i} p_j \pi_i^j, \quad i = 1, 2, \dots, j, \quad j = 1, 2, \dots, n - s$$

where π_i^j is as in the game where all quantity choices are made simultaneously. It is easy to show that for each $j = 1, 2, \dots, n - s$, the reaction functions are given by

$$q_i^j(q_s) = \frac{a - q_s - c}{1 + j}, \quad i = 1, 2, \dots, j \quad (21)$$

Plugging the above in (5) and maximizing over q_s gives us

$$q_s = \frac{a - c}{2} \quad (22)$$

Notice that q_s does not depend on $f_{n,s}^\lambda(j)$. Using (22) we have

$$q_i^{\mathcal{P}_j} = \frac{a - c}{2(1 + j)}, \quad i = 1, 2, \dots, j, \quad j = 1, 2, \dots, n - s \quad (23)$$

Using (22), (23) and (5) we obtain the value function

$$v^{seq}(S) = (a - c)^2 \sum_{j=0}^{n-s} \frac{f_{n,s}^\lambda(j)}{4(j + 1)} \quad (24)$$

Hence our game under sequential quantity choices is the pair (N, v^{seq}) .

Proposition 3 *The core of (N, v^{seq}) is empty.*

Proof We shall show that a singleton coalition $S = \{i\}$ has incentive to block an allocation. To do so it suffices to show $v^{seq}(\{i\}) > v(N)/n$. This holds if and only if

$$\sum_{j=1}^{n-1} \frac{f_{n,1}^\lambda(j)}{1 + j} > \frac{1}{n}$$

or

$$n \sum_{j=1}^{n-1} \frac{f_{n,1}^\lambda(j)}{1+j} > 1 \quad (25)$$

Recall that $\sum_{j=1}^{n-1} f_{n,1}^\lambda(j) = 1$. Moreover, for all $j = 1, 2, \dots, n-1$, we have $n \geq 1+j$. Hence

$$1 = \sum_{j=1}^{n-1} f_{n,1}^\lambda(j) < \frac{n \cdot f_{n,1}^\lambda(1)}{1+1} + \frac{n \cdot f_{n,1}^\lambda(2)}{1+2} + \dots + \frac{n \cdot f_{n,1}^\lambda(n-1)}{1+n-1} \quad (26)$$

which proves (25). ■

When S plays as a Stackelberg leader, it chooses a large quantity (relatively to the game of simultaneous play). Since we deal with a market with substitute goods, the followers will be thus induced to produce low quantities. Hence, the expected profit of S will be sufficiently high, for any S . Given that we deal with the specific case of a singleton coalition, $S = \{i\}$, the individual expected profit is high too and $S = \{i\}$ will prefer not to cooperate with N/S .

5 Conclusions

This paper analyzed a number of cooperative oligopoly games. The analysis is based on the assertion that when a coalition contemplates a deviation from the grand coalition, it assigns various non-equilibrium distributions on the set of coalition structures that the outsiders can form. This assumption is justified by imposing cognitive constraints on behalf of the firms in the market. Provided that all coalitions make simultaneous quantity choices, the corresponding games have non-empty core for sufficiently large markets.

Let us mention a few extensions of the current work. The analysis of oligopolistic markets with general demand and cost functions and/or other modes of competition (e.g., product differentiation, price competition) are natural future directions. Further, the application of the current framework to other economic environments (e.g, environmental agreements, etc.) is of interest.

References

1. Anderson S., J. K. Goeree and C. A. Holt (2002). The logit equilibrium: A perspective on intuitive behavioral anomalies, *Southern Economic Journal*, 69, 21-47.
2. Aumann, R. (1959). Acceptable points in general cooperative n-person games, *Contributions to the theory of games IV*, *Annals of Mathematics Studies* vol. 40, Princeton University Press, Princeton.

3. Bell E. T. (1934). Exponential Numbers. *American Mathematical Monthly*, 41, 411-419.
4. Camerer C. (2003). Behavioral studies of strategic thinking in games, *TRENDS in Cognitive Sciences*, 7, 225-231.
5. Camerer C., T-H Ho and J-K Chong (2004) A Cognitive hierarchy model of games, *Quarterly Journal of Economics*, 119, 861-898.
6. Chander, P. (2009). Cores of games with positive externalities, Working Paper.
7. Chander, P. and H. Tulkens (1997). A core of an economy with multilateral environmental externalities, *International Journal of Game Theory*, 26, 379-401.
8. Chen, H.C, J. Friedman and J.-F. Thisse (1997). Boundedly rational Nash equilibrium: a probabilistic choice approach. *Games and Economic Behavior*, 18, 32-54.
9. Currarini S. and M. Marini (1998). The core of games with Stackelberg leaders, Working Paper, MPRA.
10. Dang, V. D. and Jennings, N. R. (2004). Generating coalition structures with finite bound from the optimal guarantees. In: 3rd International Conference on Autonomous Agents and Multi-Agent Systems, New York, USA, 564-571.
11. Dick, A. (1996). When are cartels stable contracts? *Journal of Law and Economics*, 39, 241-283.
12. Haruvy E. and D. Stahl (2007). Equilibrium selection and bounded rationality in symmetric normal form games, *Journal of Economic Behavior and Organization*, 62, 98-119.
13. Haucap J., U. Heimeshoff and L.M Schultz (2010). Legal and illegal cartels in Germany between 1958-2004, Working Paper, Dusseldorf Institute for Competition Economics.
14. Huang, C.Y. and T. Sjostrom (1998). The p -core, Working Paper.
15. Huang, C.Y. and T. Sjostrom (2003). Consistent solutions for cooperative games with externalities, *Games and Economic Behavior*, 43, 196-213.
16. Koczy L. (2007). A recursive core for partition function form games, *Theory and Decision*, 63, 41-51.
17. McKelvey, R. and T. R. Palfrey (1995). Quantal response equilibria for normal form games, *Games and Economic Behavior* 10, 6-38.
18. Motta, M. (2009). Cartels in the European Union: Economics, Law, Practice, in Xavier Vives (ed.), *Competition in the EU: Fifty Years on from the Treaty of Rome*, Oxford University Press.
19. Rajan R. (1989). Endogenous coalition formation in cooperative oligopolies, *International Economic Review*, 30, 4, 863-876.

20. Ray D. and R. Vohra (1999). A theory of endogenous coalition structures, Games and Economic Behavior, 26, 286-336.
21. Sandholm, T., K. Larson, M. Andersson, O. Shehory, F. Tohmé (1999). Coalition structure generation with worst case guarantees. Artificial Intelligence 111, 209-238.
22. Stahl D. and P.Wilson (1994). On players' models of other players: Theory and experimental evidence, Games and Economic Behavior, 10, 218-254.
23. Zhao, J. (1999). A β -core existence result and its application to oligopoly markets, Games and Economic Behavior, 27, 153-168.

Appendix

s	$\frac{v(s)}{(a-c)^2}, \lambda = 0$	$\frac{v(s)}{(a-c)^2}, \lambda = 1$
1	0.02255	0.02261
2	0.02523	0.02530
3	0.02851	0.02858
4	0.03259	0.03266
5	0.03777	0.03785
6	0.04457	0.04462
7	0.05386	0.05386
8	0.06721	0.06712
9	0.08650	0.08633
10	0.11111	0.11111
11	0.25	0.25

Table 1: values $v(S)$ in (N, v) with $n = 11$ players.

n	$\frac{v(\{i\})}{(a-c)^2}, \lambda = 0$	$\frac{v(\{i\})}{(a-c)^2}, \lambda = 1$
3	0.08650	0.08633
4	0.06721	0.06712
5	0.05386	0.05386
6	0.04457	0.04463
7	0.03777	0.03785
8	0.03259	0.03266
9	0.02851	0.02858
10	0.02523	0.02530

Table 2: values $v(\{i\})$, $n \in \{3, 4, \dots, 10\}$